A Non-Commutative Bayes Theorem

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Classical Bayes

Probability of B given A

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Probability of A given B.

Marginal Probabilities.

Classical Bayes

$$\underbrace{P(A|B)}_{\text{Posterior}} = \frac{\overbrace{P(B|A)}^{\text{Support } B \text{ provides for } A.}}{P(B)} \underbrace{P(A)}_{\text{Prior - initial belief in } A.}$$

Idea: We will encode this diagrammatically in categories that abstract the relevant probablistic notions. This will include C^*-algebras.

FinStoch

Objects: Finite Sets X

Morphisms: Stochastic Maps $X \overset{f}{\rightsquigarrow} Y$

- associates a probability measure f_x on Y to each $x \in X$.
- $f_x(A) = \sum_{y \in A} f_{yx}$; $f_{yx} = f_x(y)$

Composition: $(g \circ f)_{zx} := \sum_{y \in Y} g_{zy} f_{yx}$

Definition: A Markov Category is a symmetric monoidal category in which every object $X \in C$ has two maps $\Delta_X : X \longrightarrow X \otimes X$ and $!X : X \longrightarrow \{\bullet\}$, called copy and discard.

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$$a_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$

$$\rho_x : X \otimes 1 \longrightarrow X$$

$$\lambda_x : 1 \otimes X \longrightarrow X$$

$$B_{xy} : x \otimes y \longrightarrow y \otimes x$$

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Satisfying relationships like:

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "a_{x,1,y}"] \arrow[dr, "\rho_x \otimes 1_y"'] & & x \otimes (1 \otimes y) \arrow[dl, "1_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}$$

and others!

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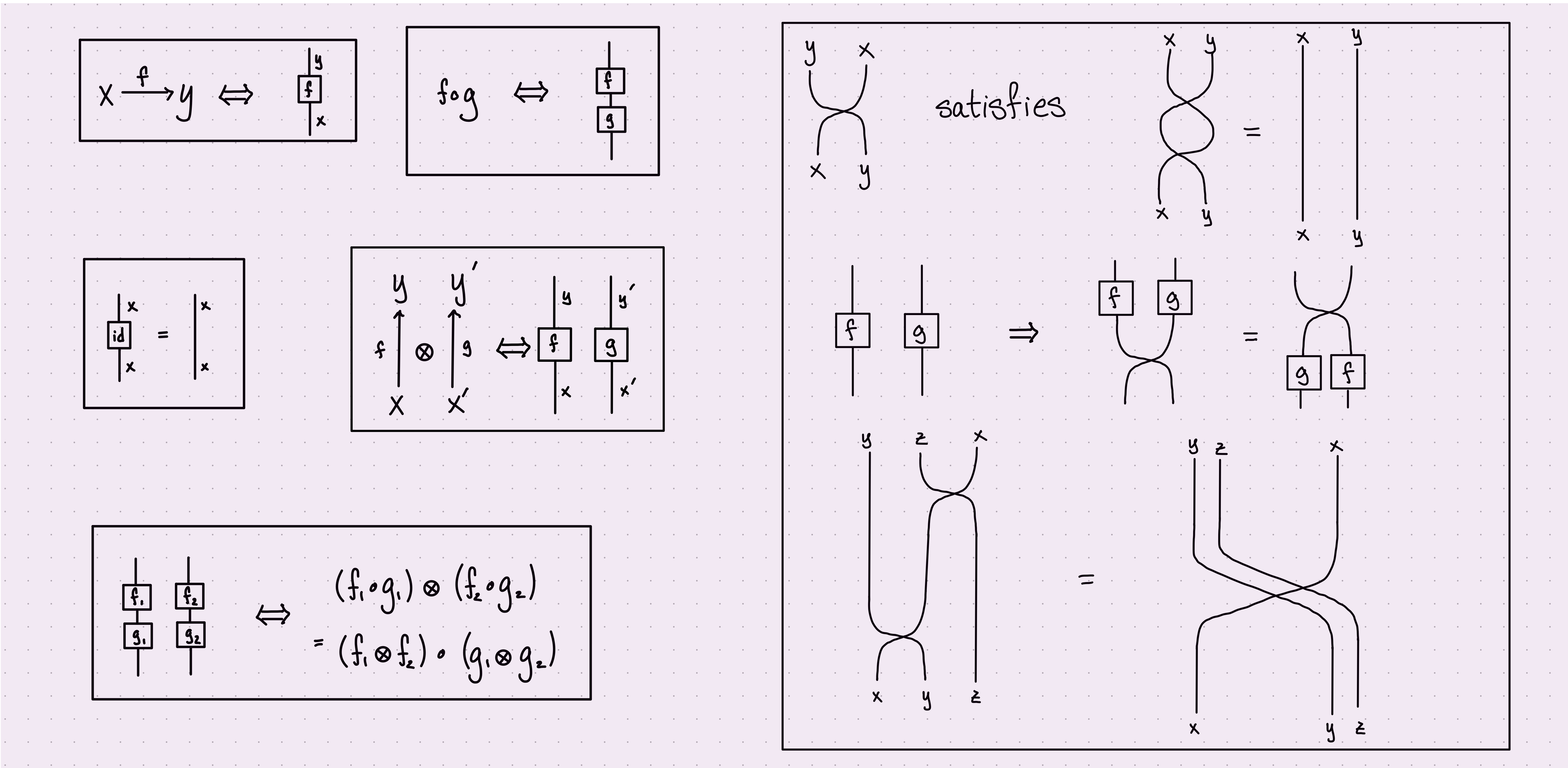

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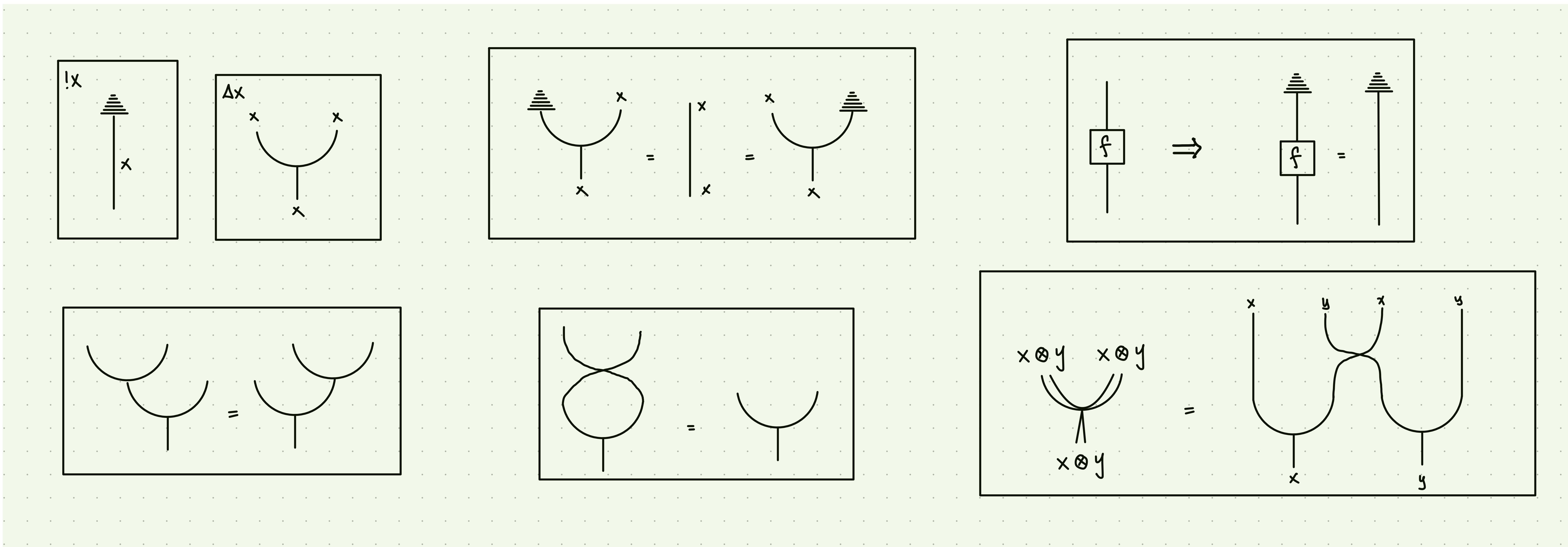


Bayes Redux

Let X and Y be finite sets. Given a probability measure p and a stochastic map $f: X \rightsquigarrow Y$, there exists a stochastic map $g: Y \rightsquigarrow X$ such that the following diagram commutes.

$$\begin{tikzcd}
Y \arrow[d, "\Delta Y"'] & \{\bullet\} \arrow[l, rightsquigarrow, "q = f \circ p"'] \arrow[r, rightsquigarrow, "p"] & X \arrow[d, "\Delta X"] \\
Y \otimes Y \arrow[r, rightsquigarrow, "g \times id_Y"'] & X \otimes Y & X \otimes X \arrow[l, rightsquigarrow, "id_X \times f"]
\end{tikzcd}$$

Bayes Redux

$$
\begin{tikzcd}
Y \arrow[d, "\Delta Y"'] & & \{\bullet\} \arrow[ll, squiggly, "q = f \circ p"'] \arrow[rr, squiggly, "p"] & & X \arrow[d, "\Delta X"] \\
Y \otimes Y \arrow[rr, squiggly, "g \times \mathrm{id}_Y"'] & & X \otimes Y & & X \otimes X \arrow[ll, squiggly, "\mathrm{id}_X \times f"]
\end{tikzcd}
$$

$$g : Y \rightsquigarrow X \qquad q : \{\bullet\} \rightsquigarrow Y$$

$$g_y \in \mathrm{Prob}(X) \qquad q \in \mathrm{Prob}(Y)$$

$$P(A|B) \cdot P(B) = \sum_{y \in B} g_y(A)\, q_y$$

$$=$$

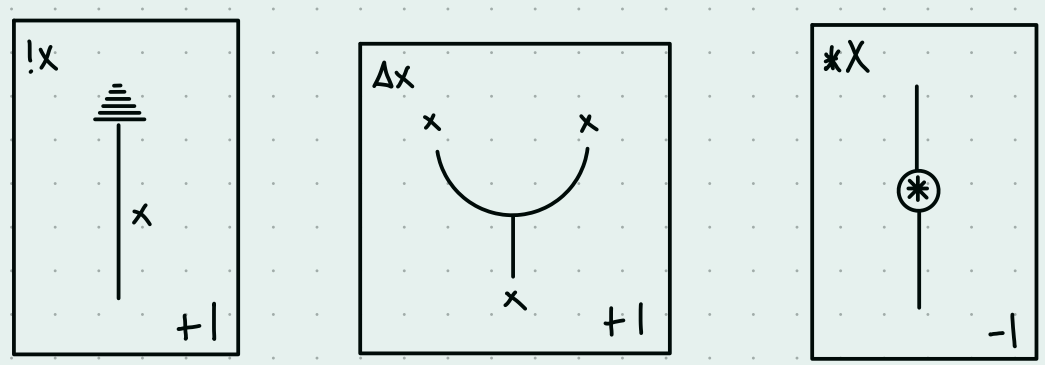
$$f_x : X \rightsquigarrow Y \qquad p : \{\bullet\} \rightsquigarrow X$$

$$f_x \in \mathrm{Prob}(Y) \qquad p \in \mathrm{Prob}(X)$$

$$P(B|A) \cdot P(A) = \sum_{x \in A} f_x(B)\, p_x$$

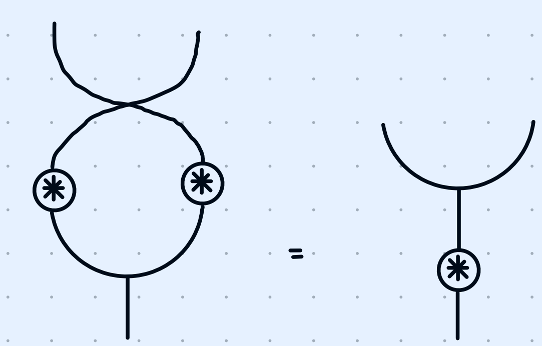
Quantum Markov Categories

Morally, these are Markov categories with an involution. However, each morphism is labeled and only morphisms with the same label can be tensored.

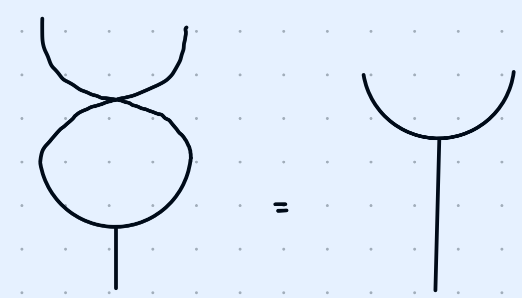


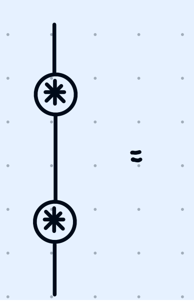
Mostly, the same rules as before.

but,



replaces





and some additional axioms.

Ex// Finite dimensional C^*-algebras with linear and conj. linear maps.
NonEx// Finite dimensional C^*-algebras with completely positive unital maps.

Bayes Redux

Let $B \overset{F}{\rightsquigarrow} A$ be a completely positive unital map between finite dimensional C^*-algebras, $A \overset{\omega}{\rightsquigarrow} \mathbb{C}$ a state, and let $\xi = \omega \circ F$. A Bayesian inverse of (F, ω) is a completely positive map $A \overset{G}{\rightsquigarrow} B$ such that

$$
\begin{tikzcd}
B \otimes B \arrow[d, squiggly, "\Delta_B"'] & A \otimes B \arrow[l, squiggly, "G \otimes id_B"'] \arrow[r, squiggly, "id_A \otimes F"] & A \otimes A \arrow[d, squiggly, "\Delta_A"] \\
B \arrow[r, squiggly, "\xi"'] & \mathbb{C} & A \arrow[l, squiggly, "\omega"]
\end{tikzcd}
$$

$b \otimes b' \mapsto bb'$ $\qquad$ $a \otimes a' \mapsto aa'$

Any linear map satisfying the above will be called a Bayes map.

(Parzygnat-R)

Proposition: Let $F : M_n \rightsquigarrow M_m$ be a completely positive unital map, $\omega = \mathrm{tr}(\rho\,\_)$, $\omega : M_m \rightsquigarrow \mathbb{C}$, and set $\xi = \omega \circ F = \mathrm{tr}(\sigma\,\_)$. Let P_ξ be the support of ξ. A Bayes map $G : M_m \rightsquigarrow M_n$ must satisfy

$$P_\xi G(A) = \hat{\sigma} F^*(\rho A).$$

- Here, we make no claims on positivity.
- However, $P_\xi G(\ ) P_\xi$ is completely positive iff $P_\xi G(\ ) P_\xi$ is $*$-preserving.

Schur Complements

$$M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \quad \text{iff} \quad \begin{array}{l} 1)\ A \geq 0 \\ 2)\ \mathrm{Ker}(A) \subseteq \mathrm{Ker}(B^*) \\ 3)\ C - B^*\hat{A}B \geq 0 \end{array}$$

Furthermore, when $M \geq 0$,

$$M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} A^{1/2} & 0 \\ B^*A^{1/2} & C - B^*\hat{A}B \end{bmatrix} \begin{bmatrix} A^{1/2} & 0 \\ B^*A^{1/2} & C - B^*\hat{A}B \end{bmatrix}^*$$

Choi-Matrix

Let $\Phi : M_n \longrightarrow M_m$ be a linear map and $C_\Phi = \sum_{ij} E_{ij} \otimes \Phi(E_{ij})$ be called the Choi matrix for Φ. The map Φ is completely positive iff C_Φ is a positive matrix.

- We can use this and the previous slide to build Bayes inverses.

(Parzygnat-R.)

Theorem: Let $F : M_n \rightsquigarrow M_m$ be a completely positive unital map, $\omega = tr(\rho\,\_)$, $\omega : M_m \rightsquigarrow \mathbb{C}$, and set $\xi := \omega \circ F = tr(\sigma\,\_)$. Let P_ξ be the support of ξ.

Set,

$$A := \sum_{i,j=1}^{m} E_{ij} \otimes \hat{\sigma} F^*(\rho E_{ij}) P_\xi \quad \text{and} \quad B = \sum_{i,j=1}^{m} E_{ij} \otimes \hat{\sigma} F^*(\rho E_{ij}) P_\xi^\perp$$

Then (F, ω) has a Bayesian inverse iff

$$A = A^* \quad \text{and} \quad tr_{M_m}(B^* \hat{A} B) \leq P_\xi^\perp.$$

Inverses, disintegrations, and Bayesian inversion in quantum Markov categories

Arthur J. Parzygnat

arXiv: 2001.08375v3

A non-commutative Bayes' theorem

Arthur J. Parzygnat and Benjamin P. Russo

arXiv: 2005.03886v1

Non-commutative disintegrations: existence and uniqueness in finite dimensions

Arthur J. Parzygnat and Benjamin P. Russo

arXiv: 1907.09689v1

A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics

Tobias Fritz

arXiv: 1908.07021v8